

A STUDY IN LOCALLY COMPACT GROUPS —CHABAUTY SPACE, SYLOW THEORY, THE SCHUR-ZASSENHAUS FORMALISM, THE PRIME GRAPH FOR NEAR ABELIAN GROUPS

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Dedicated to Professor Herbert Heyer on the occasion of his eightieth birthday

ABSTRACT. The class of *locally compact near abelian groups* is introduced and investigated as a class of metabelian groups formalizing and applying the concept of scalar multiplication. The structure of locally compact near abelian groups and its close connections to prime number theory are discussed and elucidated by graph theoretical tools. These investigations require a thorough re-viewing and extension to present circumstances of various aspects of the general theory of locally compact groups such as

- the Chabauty space of closed subgroups with its natural compact Hausdorff topology,
- a very general Sylow subgroup theory for periodic groups including their Hall systems,
- the scalar automorphisms of locally compact abelian groups,
- the theory of products of closed subgroups and their relation to semidirect products, and
- inductively monothetic groups are introduced and classified.

As applications, firstly, a complete classification is given of *locally compact topologically quasihamiltonian groups*, which has been initiated by F. Kümlich, and, secondly, Yu. Mukhin’s classification of *locally compact topologically modular groups* is retrieved and further illuminated.

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1. BACKGROUND

In this survey we describe what may be called a structure theory of locally compact near-abelian groups. This attempt becomes clearer after we describe the historical development of researching locally compact groups in a broad sense.

Keys to our understanding will be the concept of an inductively monothetic locally compact group, the Chabauty space associated canonically with a locally compact group, and a Sylow theory for closed subgroups of periodic locally compact groups reflecting typical features of the Sylow theory of finite groups. The emergence of the role of the prime numbers attached to the building blocks of the groups we consider points into the direction of graph theory which we shall employ rather extensively. However, as interesting as the structure theory of near abelian groups itself is, we emphasise here, that the machinery for developing and detailing it should share center stage.

1.1. Some History. The structure theory of locally compact groups has a long history. One of its roots is DAVID HILBERT's question of 1900 whether a locally euclidean topological group might possibly support the introduction of a differentiable parametrisation such that the group operations are in fact differentiable.

A first affirmative answer was given for *compact* locally euclidean groups when they were found to be real matrix groups as a consequence of the foundational work by HERMANN WEYL and his student FRITZ PETER in 1927 on the representation theory of compact groups. A second step was achieved when the answer was found to be “yes” for *commutative* locally compact groups. This emerged out of the fundamental duality theorems by LEV SEMYONOVICH PONTRYAGIN (1934) and EGBERT VAN KAMPEN (1937). This duality forever determined the structure and harmonic analysis of locally compact abelian groups after it was widely read in the book of 1938 and 1953 by ANDRÉ WEIL on the integration in locally compact groups [40].

A final positive answer to Hilbert's Fifth Problem had to wait almost another two decades when, 17 years after the Second World War, the contributions of ANDREW MATTEI GLEASON, DEANE MONTGOMERY, and LEO ZIPPIN around 1952 provided the final affirmative answer to HILBERT's problem. It led almost at once to the fundamental insights of HIDEHIKO YAMABE 1953, completing the pioneering work of KENKICHI IWASAWA (1949) providing the fundamental structure of all those locally compact groups G which had a compact space

G/G_0 of connected components: Such groups were recognised as being approximated by quotient groups G/N modulo arbitrarily small compact normal subgroups N in such a fashion that each G/N is a *Lie group*, that is, one of those groups on which Hilbert had focussed in the fifth of his 23 influential problems in 1900 and which SOPHUS MARIUS LIE (1842-1899) had invented together with an ingenious algebraisation method, long known nowadays under the name of Lie algebra theory. (S. [28], [23]) A special case arises when all G/N are discrete finite groups; in this case G is called *profinite*.

The solution of Hilbert's Fifth Problem in the middle of last century opened up the access to the structure theory of locally compact groups to the extent they could be approximated by Lie groups, due to the rich Lie theory meanwhile developed in algebra, geometry, and functional analysis. Recently, interest in HILBERT'S FIFTH Problem was rekindled in the present century under the influence of TERENCE TAO [39].

The quest for a solution to Hilbert's Fifth problem, at any rate, led to one major direction in the research of topological groups: in focus was the class of groups G approximable by Lie group quotients G/N , and finally G itself needed no longer to be locally compact. Such groups were called *pro-Lie groups* considered for the sake of their own. (See [15], [16].) Their theory reached as far as *almost connected* locally compact groups go (that is, those for which G/G_0 is compact), including all compact and all connected locally compact groups. But not further.

Still, every locally compact group (and every pro-Lie group) G has canonically and functorially attached to it a (frequently infinite dimensional) Lie algebra $\mathfrak{L}(G)$ and therefore a cardinal ≥ 0 attached to it, namely, its topological dimension $\text{DIM } G$ (cf. e.g, [14], 9.54, p. 498ff.). Indeed, we have the following information right away:

Proposition 1.1. *Let G be a topological group which is locally compact or a pro-Lie group. Then the following statements are equivalent:*

- (1) *Every connected component of G is singleton, that is, G is totally disconnected.*
- (2) *$\text{DIM } G = 0$, that is, G is zero-dimensional.*

If G is compact, then it is zero-dimensional if and only if it is profinite.

For this survey it is important to see that, in the 20th century, there was a second trend on the study of locally compact groups that is

equally significant even though it is opposite to the concept of connectivity in topological groups. This trend is represented by the class of compact or locally compact zero-dimensional groups.

Such groups were encountered in field theory at an early stage. Indeed, in Galois theory the consideration of the appropriate infinite ascending family of finite Galois extensions and, finally, its union would, dually, lead to an inverse family of finite Galois groups and, in the end, to their projective limit. Thus was produced what became known as a profinite group. The Galois group of the infinite field extension, equipped with the *Krull topology*, is thus a profinite group. One recognised soon that profinite groups and compact totally disconnected groups were one and the same mathematical object, expressed algebraically on the one hand and topologically on the other. Comprehensive literature on this class of groups appeared much later than text books on topological groups in which connected components played a leading role. Just before the end of the 20th century totally disconnected compact groups were the protagonists of books simultaneously entitled “Profinite Groups” by JOHN STUART WILSON and LUIS RIBES jointly with PAVEL ZALESSKII in 1998, while GEORGE WILLIS in 1994 laid the foundations of a general structure theory of totally disconnected locally compact groups if no additional algebraic information about them is available.

On the other hand, in the realm of locally compact abelian groups, the completion of the field \mathbb{Q} of rational numbers with respect to any *nonarchimedian* valuation yields the locally compact p -adic fields \mathbb{Q}_p as a totally disconnected counterpart of the connected field \mathbb{R} of real numbers. The fields \mathbb{Q}_p and their integral subrings \mathbb{Z}_p were basic building blocks of ever so many totally disconnected groups, in particular the linear groups over these field and indeed all p -adic Lie groups which NICOLAS BOURBAKI judiciously included in his comprehensive treatise on Lie groups. BOURBAKI’s text on Lie groups formed the culmination and certainly the endpoint of his encyclopaedic project extending over several decades.

The world of totally disconnected locally compact groups developed its own existence, methods and philosophies, partly deriving from finite group theory via approximation through the formation of projective limits, partly through graph theory where the appropriate automorphism theory provides the fitting representation theory, and partly also through the general impact of algebraic number theory which in the text book literature is indicated by the books of HELMUT HASSE since the thirties and lastly, 1967, by the “Basic Number Theory” of ANDRÉ WEIL, whose book on the integration on locally compact groups of

1939/1953 had influenced the progress of harmonic analysis of locally compact groups so much.

We pointed out that each locally compact group G , irrespective of any structural assumption has attached to it a (topological) Lie algebra $\mathfrak{L}(G)$ (and therefore a universal dimension). The more recent interest in zero-dimensional locally compact groups G has led to a new focus on another functorially attached invariant, namely, a compact Hausdorff space $\mathcal{SUB}(G)$ consisting of all closed subgroups of G endowed with a suitable topology and now frequently called the *Chabauty space* of G . This tool is not exactly new, but has been widely utilised in applications recently. The Chabauty space is a special case of what has been called the *hyperspace* of a compact (or locally compact) space first introduced by VIETORIS (s. [6]) In topological algebra hyperspaces were used and described e.g. in [2], [17], [18], and in all recent publications where the name of Chabauty appears in the title (e.g. [7], [8], [9]).

Notation. We shall make use of notation coherent with the book of Hofmann and Morris, [14]. The cyclic group of order p^n is denoted by $\mathbb{Z}(p^n)$ and $\mathbb{Z}(p^0)$ stands for the trivial abelian group $\{0\}$.

2. INTRODUCTORY DEFINITIONS AND RESULTS

We now return to the second thrust of the study of locally compact groups which is concerned with the research of 0-dimensional groups.

Definition 2.1. A topological group G is called *periodic* if

- (i) G is locally compact and totally disconnected
- (ii) $\overline{\langle g \rangle}$ is compact for all $g \in G$.

So a compact group is periodic if and only if it is profinite. A very significant portion of the locally compact groups considered here will be *periodic* groups. That is, we deal with totally disconnected locally compact groups in which every element is contained in a profinite subgroup. We shall say that a topological group G is *compactly ruled* if it is the directed union of its compact open subgroups. *If G is a locally compact solvable group in which every element is contained in a compact subgroup, then it is compactly ruled.* The class of *compactly ruled groups* comprises both, the class of *profinite groups* and the one of *locally finite groups*, i.e. groups where every finite subset generates a finite subgroup only, see [24]. In many cases we assume that the periodic groups we consider are compactly ruled. These properties make them topologically special; just how close they make our groups to profinite groups remains to be seen in the course of this survey. A

second significant property of the groups we study is an algebraic one: they are solvable, indeed metabelian. Again, it is another challenge to discern just how close this makes them to abelian groups. The groups we study will be called *near abelian*.

In order to offer a precise definition of the class of locally compact groups we need one preliminary definition, extending a very familiar concept:

Definition 2.2. A topological group G will be called *monothetic* if $G = \overline{\langle g \rangle}$ for some $g \in G$, and *inductively monothetic* if for every finite subset $F \subseteq G$ there is an element $g \in G$ such that $\overline{\langle F \rangle} = \overline{\langle g \rangle}$.

We shall discuss and classify inductively monothetic locally compact groups in greater detail later; but let us observe here right away a connected example illustrating the two definitions: Indeed let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the (additively written) circle group. Then

the 2-torus is monothetic but is not inductively monothetic, since $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^2 \subseteq \mathbb{T}^2$ is finitely generated but is not monothetic.

Yet in the domain of totally disconnected locally compact groups *every 0-dimensional monothetic group is inductively monothetic.*

In a periodic group, each monothetic subgroup $\overline{\langle g \rangle}$ is compact, equivalently, procyclic.

Now we are prepared for a definition of the class of locally compact groups whose details we shall consider here:

Definition 2.3. A topological group G is *near abelian* provided it is locally compact and contains a closed abelian subgroup A such that

- (1) G/A is an abelian inductively monothetic group, and
- (2) every closed subgroup of A is normal in G .

The subgroup A we shall call a *base* for G .

When we eventually collect applications for this class of locally compact groups, then we shall see that for instance all locally compact groups in which two closed subgroups commute setwise form a subclass of the class of near abelian groups and that the class of all locally compact groups in which the lattice of closed subgroups is modular is likewise a subclass of the class of near abelian groups.

2.1. Some History of Near Abelian groups. In the world of discrete groups, near abelian groups historically appeared in a natural way when K. IWASAWA attempted the classification what is now known as *quasihamiltonian* and *modular* groups as expounded in the monograph by R. SCHMIDT, [37]). It was F. KÜMMICH (cf. [25]) who initiated

in his dissertation written under the direction of PETER PLAUMANN and in papers developed from his thesis the study of *topologically quasi-hamiltonian groups*. These are topological groups such that $\overline{XY} = \overline{YX}$ is valid for any closed subgroups X and Y of such a group. A bit later YU. MUKHIN turned to investigating the class of locally compact topologically modular groups (cf. e.g. [30, 33]).

The properties that there be a closed normal abelian subgroup A of G such that G/A is inductively monothetic and such that every closed subgroup of A is normal in G suggest themselves by the fact, proved by K. IWASAWA, that discrete quasihamiltonian groups satisfy them. In a similar vein, MUKHIN, during his work on classifying topologically modular groups, finds that these groups are all near abelian in our sense (see e.g. [31]).

An earlier article by K. H. HOFMANN and F. G. RUSSO, was devoted to *classifying* compact p -groups that are topologically quasihamiltonian (cf. [19]). The major result states that such groups are at the same time topologically quasihamiltonian and near abelian with the exception of $p = 2$ in which case some sporadic near abelian groups are topologically quasihamiltonian while the bulk of them are not. This once again is evidence of the fact that is often quoted by number theorists and group theorists alike that 2 is the oddest of all primes.

In linear algebra, a group G of $(n + 1) \times (n + 1)$ -matrices of the form

$$\begin{pmatrix} r \cdot E_n & v \\ 0 & 1 \end{pmatrix}, \quad 0 < r \in \mathbb{R}, \quad v \in \mathbb{R}^n$$

with the identity E_n of $\mathrm{GL}(n, \mathbb{R})$ is a metabelian Lie group that has been called *almost abelian* (s. [12], p. 408, Example V.4.13). The subgroup A of all matrices with $r = 1$ is isomorphic to \mathbb{R}^n and every vector subspace of A is normal in G and $G/A \cong \mathbb{R}$ is a one-dimensional Lie group which is not inductively monothetic, but we shall see that inductively monothetic groups are in some sense “rank one” group analogs.

In both cases we have a representation $\psi: G/A \rightarrow \mathrm{Aut}(A)$ such that $\psi(gA)(a) = gag^{-1}$ as an essential element of structure. In the near abelian case we shall say that G is *A-nontrivial* if the image of ψ has more than 2 elements. Whereas in the Lie group case, the structure of an almost abelian Lie group G is comparatively simple, in the case of a group G satisfying the conditions of Definition 2.3 it is likely to be rather sophisticated as we illustrate by a result (s. [10], Theorem 7.4) in which $C_G(A) = \ker \psi$ denotes the centraliser $\{g \in G : (\forall a \in A) ag = ga\}$ of A in G :

Theorem 2.4. (Structure Theorem I on Near Abelian Groups) *Let G be an A -nontrivial near abelian group. Then*

- (1) *A is periodic.*
- (2) *G is totally disconnected.*
- (3) *When $\psi(G/A)$ is compact or A is an open subgroup, then G has arbitrarily small compact open normal subgroups, that is, G is pro-discrete.*
- (4) *G itself is periodic if and only if G/A is periodic if and only if G/A is not isomorphic to a subgroup of the discrete group \mathbb{Q} of rational numbers.*
- (5) *$C_G(A)$ is an abelian normal subgroup containing A and is maximal for this property.*

This shows that for our topic, *periodic* locally compact groups play a significant role.

3. INDUCTIVELY MONOTHETIC GROUPS

A good understanding of near-abelian groups depends on a clear insight into the concept of inductively monothetic groups. They were recently featured in [9].

We must recall the concept of a *local product* of a family of topological groups which in the theory of locally compact groups mediates between the idea of a Tychonoff product of compact groups and the idea of a direct sum of a family of discrete groups; the principal applications are in the domain of abelian groups, but the concept as such has nothing to do with commutativity.

Definition 3.1. Let $(G_j)_{j \in J}$ be a family of locally compact groups and assume that for each $j \in J$ the group G_j contains a compact open subgroup C_j . Let P be the subgroup of the cartesian product of the G_j containing exactly those J -tuples $(g_j)_{j \in J}$ of elements $g_j \in G_j$ for which the set $\{j \in J : g_j \notin C_j\}$ is finite. Then P contains the cartesian product $C := \prod_{j \in J} C_j$ which is a compact topological group with respect to the Tychonoff topology. The group P has a unique group topology with respect to which C is an open subgroup. Now the *local product* of the family $((G_j, C_j))_{j \in J}$ is the group P with this topology, and it is denoted by

$$P = \prod_{j \in J}^{\text{loc}} (G_j, C_j).$$

Let us note that the local product is a locally compact group with the compact open subgroup $\prod_{i \in J} C_p$. While the full product $\prod_{j \in J} G_j$

has its own product topology we note that in general the local product topology on P is properly finer than the subgroup topology. The concept of the local product was introduced and its duality theory in the commutative situation was studied by J. BRACONNIER in [3]. For us local products play a role most frequently with J being the set π of all prime numbers. This is well illustrated by the following key result on periodic locally compact abelian groups where we note, that for a locally compact abelian group G and each prime p , we have a unique characteristic subgroup G_p containing all elements g for which $\overline{\langle g \rangle}$ is a profinite p -group; G_p is called the p -primary component or the p -Sylow subgroup of G .

Theorem 3.2. (J. Braconnier) *Let G be a periodic locally compact abelian group and C any compact open subgroup of G . Then G is isomorphic to the local product*

$$\prod_{p \in \pi}^{\text{loc}} (G_p, C_p). \quad (\text{LP})$$

The following remark is useful for us as a consequence of the fact that any compact p -group C_p is a \mathbb{Z}_p -module and any prime $q \neq p$ is a unit in \mathbb{Z}_p , whence C_p is divisible by $n \in \mathbb{N}$ with $(n, p) = 1$:

Remark 3.3. A periodic locally compact abelian group G is divisible iff all p -Sylow subgroups G_p are divisible.

The structure of a locally compact *monothetic* group G is familiar to workers in the area: It is either isomorphic to the discrete group \mathbb{Z} of integers or is compact (Weil's Lemma, s. e.g. [14], Proposition 7.43, p.348.). A compact abelian group is known if its discrete Pontryagin dual is known. A compact abelian group G is monothetic if and only if there is a morphism $f: \mathbb{Z} \rightarrow G$ of locally compact groups with dense image, that is, iff there is an injection of the discrete group \widehat{G} into the character group \mathbb{T} , that is, \widehat{G} is isomorphic to a subgroup of $\mathbb{Q}^{(2^{\aleph_0})} \oplus \bigoplus_{p \in \pi} \mathbb{Z}(p^\infty)$. (Here $\mathbb{Z}(p^\infty)$, as usual, is the Prüfer group $\bigcup_{n \in \mathbb{N}} \frac{1}{p^n} \mathbb{Z} / \mathbb{Z} \subseteq \mathbb{T}$.) Whenever G is zero-dimensional, things simplify dramatically:

Proposition 3.4. *A compact zero-dimensional abelian group G is monothetic if it is isomorphic to $\prod_{p \in \pi} G_p$ where the p -factor G_p is either $\mathbb{Z}(p^m)$ for some $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ or \mathbb{Z}_p , the additive group of the ring of p -adic numbers.*

Let us proceed to inductively monothetic locally compact groups. For periodic inductively monothetic groups it is convenient to introduce

some special terminology. From Braconnier's Theorem 3.2 we know that every periodic locally compact abelian group G , for any given compact open subgroup $C \subseteq G$, is (isomorphic to) the local product

$$\prod_{p \in \pi} (G_p, C_p) \quad (*)$$

of its p -Sylow subgroups.

Definition 3.5. A topological group G is called Π -*procyclic*, if it is a periodic locally compact abelian group and each p -Sylow subgroup G_p is either a finite cyclic p -group (possibly singleton) or \mathbb{Z}_p , that is, G_p is p -procyclic.

Now we can formulate the classification of inductively monothetic locally compact groups.

Theorem 3.6. (Classification Theorem of Inductively Monothetic Groups)
Let G be an inductively monothetic locally compact group. Then G is either

- (a) *a 1-dimensional compact connected abelian group, or*
- (b) *a subgroup of the discrete group \mathbb{Q} , or*
- (c) *a periodic locally compact abelian group such that G_p is isomorphic to \mathbb{Q}_p , or $\mathbb{Z}(p^\infty)$, or \mathbb{Z}_p , or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.*

All inductively monothetic groups are sigma-compact, i.e. are countable unions of compact subsets.

The groups of connected type in Theorem 3.6 are monothetic; other types may or may not be monothetic. The periodic inductively monothetic groups G of part (b) require special attention. First we divide the set π of all prime numbers into disjoint sets

- $\pi_A = \{p \in \pi : G_p \cong \mathbb{Q}_p\}$,
- $\pi_B = \{p \in \pi : G_p \cong \mathbb{Z}(p^\infty)\}$,
- $\pi_C = \{p \in \pi : G_p \cong \mathbb{Z}_p\}$,
- $\pi_D = \{p \in \pi : (\exists n \in \mathbb{N}_0) G_p \cong \mathbb{Z}(p^n)\}$.

Now we fix a compact open subgroup C of G and identify G with the local product $\prod_{p \in \pi} (G_p, C_p)$, further we define two closed characteristic subgroups as

$$\begin{aligned} D &:= \prod_{p \in \pi_A \cup \pi_B}^{\text{loc}} (G_p, C_p), \\ P &:= \prod_{p \in \pi_C \cup \pi_D}^{\text{loc}} (G_p, C_p), \end{aligned}$$

and notice that $G = D \oplus P$. Both subgroups D and G are characteristic, and we notice that in view of Remark 3.3 D is the unique largest divisible subgroup of G

Theorem 3.7. (Classification of Inductively Monothetic Groups, continued) *Let G be a periodic inductively monothetic locally compact group. Then G is the direct topological and algebraic sum $D \oplus P$ of two characteristic closed subgroups of which D is the largest divisible subgroup of G and P is the unique largest Π -procyclic subgroup according to Definition 3.5.*

We apply this information to the structure theory of a near abelian periodic group G with base A . Then $G/A = D \oplus P$ as in Theorem 3.7; let G_D , respectively G_P denote the full inverse images for the quotient morphism $G \rightarrow G/A$. Now G_D and G_P are closed normal subgroups such that $G = G_D G_P$ and $G_D \cap G_P = A$, and we have $G_D \subseteq C_G(A)$ (see [10], Theorem 7.6).

Theorem 3.8. (Structure Theorem II on Near Abelian Groups) *Let G be a periodic near abelian locally compact group with a base A such that G is A -nontrivial. Then $A \subseteq G_D \subseteq C_G(A)$ where G_D is a normal abelian subgroup such that $G/G_D \cong G_P/A$ is Π -procyclic.*

This portion of the basic structure theory of near abelian groups in the periodic situation will allow us to concentrate largely on the case that the factor group G/A is Π -procyclic.

4. FACTORISATION AND SCALING

We begin with a definition elaborating the definition of near abelian groups.

Definition 4.1. Let G be a near abelian locally compact group with a base A . A closed subgroup H is called a *scaling subgroup* for A if

- (i) H is inductively monothetic, and
- (ii) $G = AH$.

Example 4.2. There exists a (discrete) abelian group G with a subgroup A which is not a direct summand and which has the following properties: A is the torsion subgroup of G of the form $A \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p)$ and $G/A \cong \bigcup_{n \in \mathbb{N}} \frac{1}{2 \cdot 3 \cdots p_n} \mathbb{Z} \subseteq \mathbb{Q}$.

Example 4.3. $G \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$ and there is a subgroup A which is not a direct summand such that $G/A \cong \mathbb{Z}(p^\infty)$.

In Example 4.2, the group G is a subgroup of \mathbb{R}/\mathbb{Z} and is a construction due to D. Maier [27]. Example 4.3 is inspired by Example ∇ in Theorem A1.32, p. 686 of [14].

These examples show that there are obstructions to a very *general* result asserting the existence of a scaling group for near abelian groups G with bases A .

A scaling group H , whenever it exists, is a *supplement* for A in G but not in general a *semidirect complement*. How far a supplement is from being a complement can be clarified under fairly general circumstances; we illustrate that in the following proposition.

Proposition 4.4. *Let G be a locally compact group with a closed normal subgroup A and a closed sigma-compact subgroup H containing a compact open subgroup and satisfying $G = AH$. The inner automorphisms define a morphism $\alpha: H \rightarrow \text{Aut}(A)$ by $\alpha(h)(a) = hah^{-1}$. Then we have the following conclusions:*

- (i) *The semidirect product $A \rtimes_{\alpha} H$ is a locally compact group and the function $\mu: A \rtimes_{\alpha} H \rightarrow G$, $\mu(a, h) = ah$, is a quotient morphism with kernel $\{(h^{-1}, h) : h \in A \cap H\}$ isomorphic to $A \cap H$, mapping both A and H faithfully.*
- (ii) *The factor group $G/(A \cap H)$ is a semidirect product of $A/(A \cap H)$ and $H/(A \cap H)$ and the composition*

$$A \rtimes_{\alpha} H \rightarrow G \rightarrow G/(A \cap H)$$

is equivalent to the natural quotient morphism

$$A \rtimes_{\alpha} H \rightarrow \frac{A}{A \cap H} \rtimes \frac{H}{A \cap H}$$

with kernel $(A \cap H) \times (A \cap H)$.

Notice that a scaling subgroup H of a near abelian group is sigma-compact and has a compact open subgroup, so that the proposition applies in its entirety to near abelian locally compact groups. The typical “*sandwich situation*”

$$A \rtimes H \rightarrow AH \rightarrow \frac{A}{A \cap H} \rtimes \frac{H}{A \cap H}$$

is also observed in significant ways in the structure theory of compact groups (see [14], e.g. Corollary 6.75 ff.).

So one of the most pressing questions of the structure theory of near abelian locally compact groups is the following:

Problem 1. *Under which conditions does a locally compact group G with a normal subgroup A such that G/A is inductively monothetic contain a closed inductively monothetic subgroup H such that $G = AH$?*

If G/A is in fact monothetic, then the answer is affirmative and easy. In more general circumstances we have the following theorems giving a partial answer to Problem 1.

Theorem 4.5. *Let G be a locally compact group with a compact normal subgroup A such that G/A is Π -procyclic. Then G contain a Π -procyclic subgroup H_Π such that $G = AH_\Pi$.*

Theorem 4.6. *Let G be a locally compact group with a compact open normal subgroup A such that G/A is isomorphic to an infinite subgroup of the group \mathbb{Q} . Then G contains a discrete subgroup $H \cong G/A$ such that G is a semidirect product $AH \cong A \rtimes H$.*

It would be highly desirable to have such theorems without the hypothesis that A be compact. The proofs of these theorems (see [10], Theorem 5.23ff.) make essential use of the compact Hausdorff Vietoris-Chabauty space $\mathcal{SUB}(G)$ which is attached to every locally compact group as a general invariant.

As long as this approach requires the compactness of A the following theorem may be considered as fundamental for the structure theory of near abelian locally compact groups:

Theorem 4.7. *Let G be a locally compact near abelian group with a base A such that G is A -nontrivial and G/A is Π -procyclic. Then G contains a Π -procyclic scaling subgroup H_Π for A with $G = AH_\Pi$.*

For a proof see [10], Theorem 11.3. The proof requires a wide spectrum of parts of our general structure theory of near abelian groups. In particular, at the root of this existence theorem is the Chabauty space $\mathcal{SUB}(G)$ of the group G which we mentioned earlier. This theorem and Proposition 3.8 now yield the following theorem (see [10], Theorem 11.6).

Corollary 4.8. *For every periodic locally compact near abelian group G with base A there exists a Π -procyclic closed subgroup H_Π such that $G = G_D H_\Pi$ for the abelian normal subgroup G_D with $A \subseteq G_D \subseteq C_G(A)$.*

(See [10], Theorem 6.3(iv,v) and their proofs.)

In particular, Proposition 4.4 then shows us that we have

Corollary 4.9. *Every periodic locally compact near abelian group G is a quotient of $G_D \rtimes H_\Pi$ modulo a subgroup isomorphic to $A \cap H_\Pi$.*

Recall that $Z(G)$ denotes the center of a group G .

Corollary 4.10. *For every periodic locally compact near abelian group G with base A we have $C_G(A) = AZ(G)$ and $C_G(A) \cap H_\Pi \subseteq Z(G)$, that is $AZ(G) \cap H_\Pi = Z(G) \cap H_\Pi$.*

(See [10], Theorem 6.3(iv,v) and its proof.)

The following theorem then is rather definitive on the factorisation of a periodic near abelian locally compact group and may be considered as one of the main theorems on their structure.

Theorem 4.11. (Structure Theorem III on Periodic Near Abelian Groups) *For every periodic locally compact near abelian group G with a base A such that G is not A -trivial, we have*

$$G = AZ(G)H$$

with a Π -procyclic scaling group H for A in G_P , where $AZ(G) \cap H = Z(G) \cap H$.

For information as to which closed subgroups $A^* \subseteq AZ(G)$ containing A may still be taken as base subgroups, see [10]. Theorem 10.32. The role of the center $Z(G)$ in $C_G(A) = AZ(G)$ —a locally compact abelian group we know to be a local product of its p -primary components $A_p Z(G)_p$ —is still a bit mysterious; more information will be forthcoming in Theorem 7.2 below.

5. THE SYLOW THEORY OF PERIODIC GROUPS

Sylow theory, i.e., existence and conjugacy of maximal p -subgroups, and, more generally, of maximal σ -subgroups where σ is a set of primes, is available for profinite groups (see [44, 36]). Several attempts have been made in order to generalise Sylow theory to non-compact and locally compact groups, see e.g. the survey from 1964 by Čarin, [4], or, more recently, Platonov in [34] and Reid in [35]. Here we shall focus on our class of *compactly ruled groups*. If the topology on a compactly ruled group is discrete, the group is *locally finite*, i.e., every finite subset generates a finite subgroup. Then our Sylow theory reduces to the one presented in the book of Kegel and Wehrfritz, see [24].

In each locally compact *periodic* group G the concept of a p -group can be defined meaningfully. Indeed if $g \in G$, then $M := \overline{\langle g \rangle}$ is a zero-dimensional monothetic compact group, and thus

$$M \cong \prod_{p \in \pi} M_p,$$

where for each prime p the p -primary component M_p is either $\cong \mathbb{Z}_p$ or $\mathbb{Z}(p^{n_p})$ for some $n_p = 0, 1, 2, \dots$

It is practical to generalise the concept of a p -element: For each subset $\sigma \subseteq \pi$, an element $g \in G$ is called a σ -element if $\overline{\langle g \rangle} = \prod_{p \in \sigma} M_p$; if $\sigma = \{p\}$, then g is called a p -element. The group G is a σ -group, if all of its elements are σ -elements. A subgroup S is called a σ -Sylow subgroup of G if it is a maximal element in the set of σ -subgroups. A simple application of Zorn's Lemma shows that every σ -element is contained in a σ -Sylow subgroup. We record:

Lemma 5.1. (The Closure Lemma) *Let G be any locally compact totally disconnected group. Then for any subset $\sigma \subseteq \pi$, then set G_σ of all σ -elements of G is closed in G .*

Let us look to some traditional splitting theorems that still work in the general background of periodic locally compact groups.

5.1. The Schur-Zassenhaus Splitting. The splitting of finite groups into products of subgroups of relatively prime orders can be generalized to the locally compact setting up to a point, as we show in the following. For locally finite groups the results to be discussed are well known, see e.g. [24]. They also relate to work of the second author, see [13, 17].

Proposition 5.2. *Let N be a closed subgroup of a locally compact periodic group G and assume $N \subseteq G_\sigma$. Then the following conditions are equivalent:*

- (1) N is a normal Sylow subgroup.
- (2) $N = G_\sigma$.
- (3) N is normal and G/N contains no p -element with $p \in \sigma$.

Definition 5.3. Let G be a locally compact periodic group and N a closed subgroup. We say that N satisfies the *Schur-Zassenhaus Condition* if and only if it satisfies the equivalent conditions of Proposition 5.2 for $\sigma = \pi(N)$.

Theorem 5.4. (Schur-Zassenhaus Theorem) *Let G be a periodic group and N a closed subgroup satisfying the following two conditions:*

- (1) N satisfies the Schur-Zassenhaus Condition.
- (2) G/N is a directed countable union of compact subgroups.

Then the following conclusions hold:

- (i) N possesses a complement H in G .
- (ii) Let K be a closed subgroup of G such that $K \cap N = \{1\}$ and assume that G/N is compact. Then there is a $g \in G$ such that $gKg^{-1} = H$.

It should be remarked, that for solvable groups (such as near abelian groups) the periodic groups are always directed unions of their open compact subgroups. In such a situation condition (2) simply means that G/N is sigma-compact.

The Schur-Zassenhaus configuration in the locally compact environment is delicate, since, in general problems arise with the product of a closed normal subgroup and a closed subgroup; such a product need not be closed, in general.

Still we do have theorems like the following:

Theorem 5.5. *Let N be a normal σ -Sylow subgroup of a locally compact periodic group G . Then a $(\pi \setminus \sigma)$ -Sylow subgroup H of G exists such that NH is an open and hence closed subgroup. Moreover, if H is any $(\pi \setminus \sigma)$ -Sylow subgroup of G , then NH is closed in G and H is a complement of N in NH , that is, $NH = N \rtimes H$.*

5.2. Sylow Subgroups Commuting Pairwise. Let $p \in \pi$ denote any prime and $p' := \pi \setminus \{p\}$.

Then we have the following result:

Lemma 5.6. *For a compactly ruled group G and a prime number p , the following conditions are equivalent:*

- (1) $[G_p, G_{p'}] = \{1\}$.
- (2) Both G_p and $G_{p'}$ are subgroups, and $G = G_p \times G_{p'}$.
- (3) There is a unique projection $\text{pr}_p: G \rightarrow G_p$ with kernel $G_{p'}$.

Definition 5.7. For a periodic locally compact group G we write

$$\nu(G) = \{p \in \pi : [G_p, G_{p'}] = \{1\}\}.$$

We have found the following structure theorem very useful in the context of near abelian groups generalising the well-known fact that a *pronilpotent* group is the cartesian product of its Sylow subgroups (cf. [36]):

Theorem 5.8. *In a compactly ruled locally compact group G , the set $G_{\nu(G)'} of α -elements with $\alpha \cap \nu(G) = \emptyset$ is a closed normal subgroup, and all p -Sylow subgroups for $p \in \nu(G)$ are normal subgroups. Moreover,$*

$$G \cong G_{\nu(G)'} \times \prod_{p \in \nu(G)}^{\text{loc}} (G_p, U_p)$$

for a suitable family of compact open subgroups $U_p \subseteq G_p$ as p ranges through $\nu(G)$.

5.3. The Internal Structure of Sylow Subgroups of Near Abelian

Groups. For periodic near abelian groups, to which we can apply a Sylow theory meaningfully we assume that G is a periodic near abelian locally compact group such that G is nontrivial for a base A .

Theorem 5.9. *Let G be a periodic near abelian group and A a base for which G is A -nontrivial and which satisfies $A = C_G(A)$. Then, for every set σ of prime numbers there is a σ -Sylow subgroup S_σ . Fix a σ -Sylow subgroup S_σ of G . Then*

- (i) $S_\sigma \cap A$ is the σ -primary component A_σ of A , equivalently, the σ -Sylow subgroup of A . Moreover,
- (ii) $S_\sigma/A_\sigma \cong S_\sigma A/A = (G/A)_\sigma$.
- (iii) If $(G/C_G(A))_\sigma \cong H/(H \cap C_G(A))$ is compact, then any two σ -Sylow subgroups of G are conjugate.
- (iv) $S_\sigma = C_G(A)_\sigma H_\sigma = A_\sigma Z(G)_\sigma H_\sigma$, where H is as in Corollary 4.8.

(See [10], Theorem 10.1.) The case that $\sigma = \{p\}$ is an important special case. Let us note that it may happen, $S_p \subseteq A$, in which case we have $H_p = \{1\}$.

6. SCALAR AUTOMORPHISMS

Among the methods we are using, the specification of scalar automorphisms of a periodic locally compact abelian group is prominent. Every locally compact abelian p -group A is a natural \mathbb{Z}_p -module, and in the case of a *periodic* locally compact abelian group

$$A = \prod_{p \in \pi}^{\text{loc}} (A_p, C_p) \quad (\text{LQ})$$

it is a natural

$$\tilde{\mathbb{Z}} = \prod_{p \in \pi} \mathbb{Z}_p$$

module by componentwise scalar multiplication

$$z \cdot g = (z_p)_{p \in \pi} \cdot (g_p)_{p \in \pi} = (z_p \cdot g_p)_{p \in \pi}$$

The compact ring $\tilde{\mathbb{Z}}$ is the profinite compactification of the ring \mathbb{Z} of integers.

Lemma 6.1. (The Scalar Morphism Lemma) *For a continuous automorphism α of a periodic locally compact abelian group G the following conditions are equivalent:*

- (1) $\alpha(H) \subseteq H$ for all closed subgroups H of G .

- (2) $\alpha(\overline{\langle g \rangle}) \subseteq \overline{\langle g \rangle}$ for all $g \in G$.
- (3) $\alpha(g) \in \overline{\langle g \rangle}$ for all $g \in G$.
- (4) There is an $r \in \widetilde{\mathbb{Z}}$ such that $\alpha(a) = r \cdot a$ for all $g \in G$.

We note in passing that the first three conditions are equivalent in any locally compact group.

Definition 6.2. An automorphism $\alpha \in \text{Aut}(A)$ of a periodic locally compact abelian group A is called a *scalar automorphism*. The group of all scalar automorphisms is written $\text{SAut}(A)$.

For $r \in \widetilde{\mathbb{Z}}^\times$, the group of invertible elements of $\widetilde{\mathbb{Z}}$, we denote the function $a \mapsto r \cdot a : A \rightarrow A$ by $\mu_r \in \text{SAut}(A)$.

Proposition 6.3. *Let A be a periodic locally compact abelian group. Then*

- (i) $r \mapsto \mu_r : \widetilde{\mathbb{Z}}^\times \rightarrow \text{SAut}(A)$ is a quotient morphism of compact groups. In particular, $\text{SAut}(A)$ is a profinite group and thus does not contain any nondegenerate divisible subgroups.
- (ii) *The following conditions are equivalent:*
 - (a) $\text{SAut}(A) = \{\text{id}_A, -\text{id}_A\}$,
 - (b) *The exponent of A is 2, 3, or 4.**In particular, A has exponent 2 if and only if $-\text{id}_A = \text{id}_A$.*

In the process of these discussions, we recover in our framework the following theorem of MUKHIN [31]:

Theorem 6.4. *Let A be a locally compact abelian group written additively.*

- (a) *If A is not periodic, then $\text{SAut}(A) = \{\text{id}, -\text{id}\}$.*
- (b) *If A is periodic, then $\text{SAut}(A) = \prod_p \text{SAut}(A_p)$, where $\text{SAut}(A_p)$ may be identified with the group of units of the ring of scalars of A_p :*

$$\begin{cases} \mathbb{Z}_p, & \text{if the exponent of } A_p \text{ is infinite,} \\ \mathbb{Z}_p/p^m \mathbb{Z}_p \cong \mathbb{Z}(p^m), & \text{for suitable } m \text{ otherwise.} \end{cases}$$

- (c) *In particular, $\text{SAut}(A)$ is a homomorphic image of $\widetilde{\mathbb{Z}}^\times$.*
- (d) *An automorphism α is in $\text{SAut}(A)$ iff there is a unit $z \in \widetilde{\mathbb{Z}}^\times$ such that*

$$(\forall g \in G) \alpha(g) = z \cdot g = \prod_p z_p \cdot g_p \text{ for } z = \prod_p z_p, g = \prod_p g_p.$$

The significance of Mukhin's Theorem for the structure theory of near abelian groups is visible in the very Definition 2.3 via Theorem 4.11. Indeed if G is a near abelian locally compact group with a base

A , then the inner automorphisms of G induce a faithful action of the Π -procyclic factor group $G/C_G(A) \cong H/(H \cap Z(G))$ upon the base A . So $\text{SAut}(A)$ is a quotient of H and therefore is Π -procyclic.

The structure of G is largely determined by the structure of $\text{SAut}(A)$ and therefore by the group $\tilde{\mathbb{Z}}^\times$ of units of $\tilde{\mathbb{Z}}$.

6.1. The Group of Units of the Profinite Compactification of the Ring of Integers and its Prime Graph. The group $\tilde{\mathbb{Z}}^\times$ is more complex than it appears at first. Its Sylow theory or primary decomposition is best understood in graph theoretical terms. The same graph theory turns out to be almost indispensable for dealing with the Sylow structure of near abelian groups in general. The graphs that we use are all subgraphs of a “universal” graph (which we also call the “master graph”) and which is used precisely to describe the Sylow theory of $\tilde{\mathbb{Z}}^\times$. We discuss it in the following.

A *bipartite graph* consists of two disjoint sets U and V and a binary relation $E \subseteq (U \cup V)^2$ such that $(u, v) \in E$ implies $u \in U$ and $v \in V$. The elements of $U \cup V$ are called *vertices* and the elements of E are called *edges*. Any triple (U, V, E) of this type is called a *bipartite graph*.

In the following we construct a special bipartite graph

$\mathcal{G} = (U, V, E)$ with $U, V \subseteq \mathbb{N} \times \{0, 1\}$ as follows:

Definition 6.5. Define $U = \mathbb{N} \times \{1\}$, $V = \mathbb{N} \times \{0\}$.

Let $n \mapsto p_n$ be the unique order preserving bijection of \mathbb{N} onto the set π of prime numbers. On π we consider the binary relation

$$(1) \quad T = \{(p, q) \in \pi \times \pi : q = p \text{ or } p|(q - 1)\}.$$

Let $E \subseteq (\mathbb{N} \times \{0, 1\})^2$ be defined as follows

$$(2) \quad E = \{((m, 1), (n, 0)) : (p_m, q_n) \in T\}.$$

If $e = ((m, 1), (n, 0)) \in E$ is an edge we shall use the following notation for the prime numbers associated with e :

$$p_e = p_m \quad q_e = q_n.$$

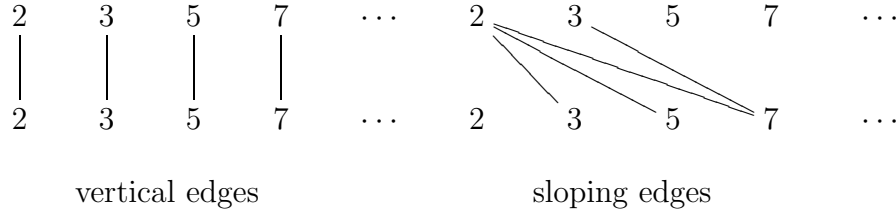
We shall call $\mathcal{G} = (U, V, E)$ the *prime master-graph*. In all bipartite graphs we consider in this text, the two sets U and V of vertices remain constant, while the set of edges will vary through subsets of E as defined in Definition 6.5.

6.2. Geometric Properties of the Master-Graph. The prime master-graph can be drawn and helps in forming a good intuition of the combinatorics involved.

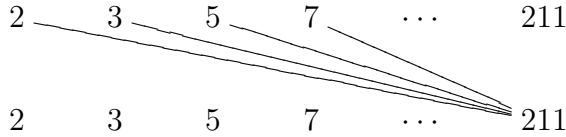
- The set of vertices $U \cup V$ of the master-graph is naturally contained in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and so we can “draw” it quite naturally.

The elements in $\mathbb{N} \times \{1\}$ are called the *upper vertices*, those in $\mathbb{N} \times \{0\}$ the *lower vertices*

- The edges $e = ((n, 1), (n, 0))$, $n \in \mathbb{N}$ are called *vertical*. All other edges $e = ((m, 1), (n, 0))$ are called *sloping*. Because of $p_m | (q_n - 1)$ they are sloping “from left-above to right-below”. There are only vertical and sloping edges. We call vertices $u = (m, 1)$ and $v = (n, 0)$ *connected* iff $e = (v, u) \in E$, i.e., if v is the upper vertex (end-point) of the edge e and v is the lower vertex (end-point) of e .



- Each lower vertex $(n, 0)$ is the endpoint of one vertical and *finitely many* sloping edges. It is connected to an upper vertex $(m, 1)$ iff $p_m | (q_n - 1)$.



Edges in U connected to the lower edge with label 211 in V .

- Each upper vertex $(m, 1)$ is connected to infinitely many lower vertices $(n, 0)$, namely, all those for which $p_m | (q_n - 1)$, that is, for which there is a natural number k for which $q_n = kp_m + 1$. Indeed, Dirichlet’s Prime Number Theorem says: *Every arithmetic progression of the form $\{ka + b : k \in \mathbb{N}\}$ with a and b relatively prime, contains infinitely many primes.*

Definition 6.6. Let p and q be any primes, say, $p = p_m$ and $q = q_n$. Then

$$\mathcal{E}_p = \{e : e = ((m, 1), (n, 0)) \in E \text{ such that } p | (q_n - 1)\},$$

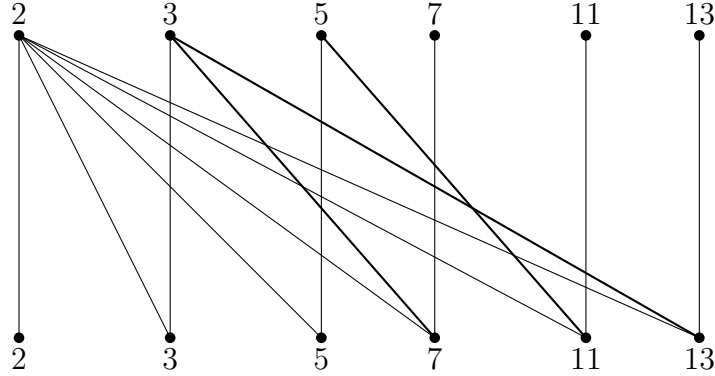


FIGURE 1. The initial part of the master graph.

the set of all edges emanating downwards from the vertex $(m, 1) \in U$ will be called the *cone peaking at p* . This cone contains infinitely many edges while

$$\mathcal{F}_q = \{e : e = ((m, 1), (n, 0)) \in E \text{ such that } p_m | (q - 1)\},$$

the set of edges ending below in the vertex $(n, 0) \in V$, called the *funnel pointing to q* , contains only finitely many edges.

All applications of prime graphs which we use in the structure theory of near abelian groups are subgraphs of this master graph.

Since for any periodic locally compact abelian group A we have a canonical surjective morphism $\mu: \tilde{\mathbb{Z}}^\times \rightarrow \text{SAut}(A)$ we need explicit information on the primary structure – or p -Sylow structure – of $\tilde{\mathbb{Z}}^\times$. We are now going to describe this structure in additive notation in terms of the prime master-graph $\mathcal{G} = (U, V, E)$.

Let $e = ((m, 1), (n, 0)) \in E$ be an edge in the master-graph.

Case 1. $m = n$. Then we set $\mathbb{S}_e = \mathbb{Z}_{p_m}$.

Case 2. $m < n$. Then $p_m | (q_n - 1)$. Assume that the $q_n - 1 = p_m^{k(e)} s(e)$ with $s(e)$ relatively prime to $p_e := p_m$. Then we set $\mathbb{S}_e = \mathbb{Z}(p_m^{k(e)})$

For the following proposition we recall

$$\tilde{\mathbb{Z}}^\times = \prod_{q \in \pi} \mathbb{Z}_q^\times.$$

Since \mathbb{Z}_q^\times is not a q -Sylow subgroup, this is *not* the q -primary decomposition of $\tilde{\mathbb{Z}}^\times$. That decomposition we describe now:

Proposition 6.7. (The Sylow Structure of $\tilde{\mathbb{Z}}^\times$) *Let $p, q \in \pi$ be primes. Then*

(i) *The structure of \mathbb{Z}_q^\times (in additive notation) is*

$$\prod_{e \in \mathcal{F}_q} \mathbb{S}_e = \mathbb{Z}_q \times \prod_{e \in \mathcal{F}_q, \text{ sloping}} \mathbb{Z}(p_e^{k(e)}).$$

(ii) *The p -primary component or p -Sylow subgroup of $\tilde{\mathbb{Z}}^\times$*

$$(\tilde{\mathbb{Z}}^\times)_p = \prod_{e \in \mathcal{E}_p} (\mathbb{Z}_{q_e}^\times)_{p_e}$$

is (in additive notation)

$$(\tilde{\mathbb{Z}}^\times)_p = \prod_{e \in \mathcal{E}_p} \mathbb{S}_e = \mathbb{Z}_p \times \prod_{e \in \mathcal{E}_p, \text{ sloping}} \mathbb{Z}(p^{k(e)}).$$

6.3. The Structure of the Invertible Scalar Multiplications of an Abelian Group and their Prime Graph.

Now let A be a periodic locally compact abelian group; the Sylow structure of $\text{SAut}(A)$ is now easily discussed: The quotient morphism $\mu: \tilde{\mathbb{Z}}^\times \rightarrow \text{SAut}(A)$, preserving the Sylow structures, and the structure of $\text{SAut}(A)$ described so far in Theorem 6.4 allow a precise description of the Sylow structure of $\text{SAut}(A)$.

We associate with A the bipartite graph $\mathcal{G}(A) = (U, V, E(A))$ with U and V as in the master-graph and with

$$E(A) = \{e \in E : e = ((m, 1), (n, 0)) \text{ such that } \text{SAut}(A_{q_n}) \neq \{\text{id}_A\}\},$$

and we define

$\mathcal{E}_p = \{e \in E(A) : e = ((m, 1), (n, 0)) \in e(A) \text{ such that } p|(q_n - 1)\}$, the set of all edges in $\mathcal{G}(A)$ ending at the vertex $(n, 0) \in V$ such that $\text{SAut}(A_{q_n})$ is nontrivial, and

$\mathcal{F}_q = \{e \in E(A) : e = ((m, 1), (n, 0)) \in E(A) \text{ such that } p_m|(q - 1)\}$, the set of all edges in $\mathcal{G}(A)$ ending at the vertex $(n, 0) \in V$ with $q_n = q$ such that $\text{SAut}(A_q)$ is nontrivial.

We recall that for each q -primary component A_q , the ring of scalars $\text{SAut}(A_q)$ is either cyclic of order q^r , the exponent of A_q , if it is finite, and is $\cong \mathbb{Z}_q$ otherwise. Thus its q -primary component is

$$\cong \begin{cases} \mathbb{Z}(q^{r-1}) & \text{if the exponent of } A_q \text{ is finite} \\ \mathbb{Z}_q & \text{otherwise.} \end{cases}$$

Accordingly we define, for each edge $e = ((m, 1), (n, 0)) \in E(A)$ in the graph $\mathcal{G}(A)$

$$\mathbb{S}_e(A) = \begin{cases} \mathbb{Z}(q_m^{r-1}) & \text{if } m = n, \text{ and } A_q \text{ has finite exponent } q^r \\ \mathbb{Z}_{q_m} & \text{if } m = n, \text{ and } A_q \text{ has infinite exponent,} \\ \mathbb{Z}(p_m^{k(e)}) & \text{if } m < n. \end{cases}$$

Then we have, analogously to Proposition 6.7, the following theorem, complementing Proposition 6.7 and Theorem 6.4:

Theorem 6.8. (The Sylow Structure of $\text{SAut}(A)$) *Let A be a periodic locally compact abelian group and $\text{SAut}(A) = \prod_{p \in \pi} \text{SAut}(A)_p$ the p -primary decomposition of the profinite group $\text{SAut}(A)$. Then*

- (i) *The p -primary decomposition of $\text{SAut}(A_q)$ is (additive notation assumed)*

$$\prod_{e \in \mathcal{F}_q} \text{SAut}(A_{q_e})_{p_e} \cong \prod_{e \in \mathcal{F}_q} \mathbb{S}_e(A) = \mathbb{Z}_q \times \prod_{e \in \mathcal{F}_q, \text{ sloping}} \mathbb{Z}(p_e^{k(e)}).$$

- (ii) *The structure of the p -primary component $\text{SAut}(A)_p$ of $\text{SAut}(A)$ (in additive notation) is*

$$\prod_{e \in \mathcal{E}_p} (\text{SAut}(A_{q_e})_{p_e}) = \prod_{e \in \mathcal{E}_p} \mathbb{S}_e(A) = \mathbb{Z}_p \times \prod_{e \in \mathcal{E}_p, \text{ sloping}} \mathbb{Z}(p_e^{k(e)}).$$

This theorem illustrates the usefulness of the prime graph $\mathcal{G}(A)$ which elucidates the fine structure of $\text{SAut}(A)$. In many instances, the prime graph is equally helpful in the discussion of the Sylow structure of any periodic near abelian locally compact group G (see [10], Section 10).

7. THE PRIME GRAPH OF A NEAR ABELIAN GROUP

Staying with a periodic near abelian group G which is A -nontrivial for a base group A , we investigate the interaction of the different Sylow subgroups in terms of the *prime graph* defined as a subgraph of the master-graph $\mathcal{G} = (U, V, E)$ of Definition 6.5 as follows:

Definition 7.1. Let G be a periodic near abelian A -nontrivial locally compact group with a base group A and write $G = AZ(G)H$.

A subgraph $\mathcal{G}_G = (U_G, V_G, E_G)$ of the master-graph \mathcal{G} is called the *prime graph* of G provided the following conditions are satisfied:

- (i) We call $(m, 1)$ the *upper p -vertex* iff $p = p_m$ and $(n, 0)$ the *lower q -vertex* iff $q = q_n$.

- (ii) An edge $e = ((m, 1), (n, 0))$ of the mastergraph is an edge in E_G if and only if $[H_{p_m}, A_{q_m}] \neq \{1\}$. With $p = p_m$ and $q = q_n$ this edge is written e_{pq} and called an *edge leading from p to q* .
- (iii) $(m, 1)$ is an upper vertex in U_G iff $(G/C_G(A))_p \neq \{1\}$.
- (iv) $(n, 0)$ is a lower vertex in V_G iff $A_p \neq \{1\}$.

We have a much sharper conclusion:

Theorem 7.2. (Structure Theorem IV on Periodic Near Abelian Groups)
Let G be a periodic A -nontrivial near abelian group. Let e_{pq} be an edge in \mathcal{G}_G .

Then we have the following conclusions (see [10], Theorem 10.13):

- (1) *If $p \neq q$, that is, e_{pq} is sloping, then $p \neq 2$ and $p|(q-1)$, but above all*
 - (C₁) *for $x \in G_p \setminus C_G(A_q)$ the function $a \mapsto [x, a] : A_q \rightarrow A_q$ is an automorphism of A_q . In particular, $[x, A_q] = A_q$.*
 - (C₂) $A_q \cap Z(G) = \{1\}$.
- (2) *If $p = q$, that is, e_{pq} is vertical, then there is a unit $s \in \mathbb{Z}_q^\times$ and a natural number $m \in \mathbb{N}$ such that $[x, a] = a^{q^m s}$ for all $a \in A_q$, that is, $[x, A_q] = A_q^{q^m}$, and $A_p \cap Z(G)$ has an exponent dividing q^m .*

The theorem gives an impression of the circumstances in which the intersection $A_q \cap Z(G)_q$ can be nontrivial: The lower q -vertex has to be isolated in the prime graph in such a case.

8. APPLICATION 1: THE CLASSIFICATION OF TOPOLOGICALLY QUASIHAMILTONIAN GROUPS

The following definition is due to F. KÜMMICH [25]:

Definition 8.1. A topological group G is called *topologically quasihamiltonian* if $\overline{XY} = \overline{YX}$ holds for any pair of closed subgroups X and Y of G .

This is equivalent to saying that \overline{XY} is a closed subgroup whenever X and Y are subgroups of G .

With the framework provided by near abelian locally compact groups, it is possible to classify completely the class of topologically quasihamiltonian locally compact groups. The classification proceeds in two steps: In a first step we classify all locally compact topologically quasihamiltonian groups, and in a second step we classify all locally compact topologically quasihamiltonian groups in one fell swoop.

For step 1 we need a definition:

Definition 8.2. The groups M_n defined by generators and relations for $n = 2, 3, \dots$ according to

$$M_n := \langle a, b \mid b^{2^n} = 1, b^{2^{n-1}} = a^2, bab^{-1} = a^{-1} \rangle$$

are called *generalised quaternion groups*.

These groups also satisfy the relations

$$a^4 = 1 \text{ and } [a, b] = a^2$$

and are fully characterised by the following explicit construction:

$$M_n \cong \frac{\mathbb{Z}(4) \rtimes \mathbb{Z}(2^n)}{\Delta},$$

where $\mathbb{Z}(2^n)$ acts on $\mathbb{Z}(4)$ by scalar multiplication with ± 1 and where Δ is generated by (s, t) , $s = 2 + 4\mathbb{Z}$ and $t = 2^{n-1} + 2^n\mathbb{Z}$. (Cf. [19], Definition 5.8.) We note that M_2 is (isomorphic to) the usual group of quaternions $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of eight elements.

Here is step 1:

Theorem 8.3. *A locally compact p -group G is topologically quasihamiltonian if and only if G is near abelian with a base group A and an inductively monothetic p -group G/A and at least one of the following statements holds:*

- (a) G is abelian.
- (b) *There is a p -procyclic scaling group $H = \overline{\langle b \rangle}$ such that $G = AH$ and there is a natural number $s \geq 1$, respectively, $s \geq 2$, if $p = 2$, such that $a^b = a^{1+p^s}$ for all a in A . The group G is A -nontrivial.*
- (c) $p = 2$ and $G \cong A_2 \times M_n$, where A_2 is an exponent 2 locally compact abelian group according, and M_n is the generalised quaternion group of order 2^{n+1} . In this case,

$$A = A_2 \times \langle a \rangle \cong \mathbb{Z}(2)^{(I_1)} \times \mathbb{Z}(2)^{I_2} \times \mathbb{Z}(4)$$

with a as in Definition 8.2 for suitable sets I_1 and I_2 . The group G is A -trivial.

Next step 2 (see [10], Theorem 13.9):

Theorem 8.4. *Let G be a locally compact periodic topologically quasihamiltonian group. Then, for each $p \in \pi(G)$, the set of p -elements G_p is a topologically quasihamiltonian p -group, and there is a compact open subgroup U_p in G_p such that $G = G_{\nu(G)}$ is (up to isomorphism) the local product of topologically quasihamiltonian p -groups*

$$G \cong \prod_{p \in \pi(G)}^{\text{loc}} (G_p, U_p).$$

Conversely, every group isomorphic to such a local product is a topologically quasiamiltonian group.

This theorem is proved with the aid of our Theorem 5.8. For nonperiodic abelian groups we give an algorithmic description of topologically quasiamiltonian locally compact groups in [10], Theorem 13.14. Except for $p = 2$ it turns out that topologically quasiamiltonian groups in the general locally compact domain are the same thing as near abelian groups. For the exceptional compact 2-groups that are near abelian but fail to be topologically quasiamiltonian see [19].

Theorem 8.4 can be visualised in terms of its prime graph, that all connected components are either vertical edges and its end points or are isolated vertices. If we allow ourselves the identification of the connected components of the prime graph with the subgroups they represent, we could reformulate Theorem 8.4 as follows:

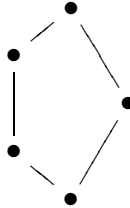
Theorem 8.5. *Let G be a locally compact periodic topologically quasiamiltonian group. Then each connected component of the prime graph of G represents a normal p -Sylow subgroup and G is a local direct product of these subgroups.*

9. APPLICATION 2: THE CLASSIFICATION OF TOPOLOGICALLY MODULAR GROUPS

Recall that the closed subgroups of a topological group form a lattice w.r.t. inclusion “ \subseteq ” as partial order.

Definition 9.1. A topological group G is called *topologically modular* if the lattice of closed subgroups is modular, that is, satisfies the law $X \vee (Y \cap Z) = (X \vee Y) \cap Z$ whenever X is a closed subgroup of Z .

This is equivalent to saying that the lattice of closed subgroups does not contain a sublattice isomorphic to



(See [37], Theorem 2.1.2.)

It is instructive to spend some time on an example due to MUKHIN which shows that topologically modular groups can be tricky.

Example 9.2. Let p be any prime and I any infinite set (e.g. $I = \mathbb{N}$), set $E := \mathbb{Z}(p)$, and define $G_j = E^2$, $C_j = \{0\} \times E$ for all $j \in I$, and set

$$G := E^{(I)} \times E^I \cong \prod_{j \in I}^{\text{loc}} (G_j, C_j),$$

where we took the discrete topology on the direct sum $E^{(I)}$ and the product topology on E^I . We shall identify G with $\prod_{j \in I}^{\text{loc}} (G_j, C_j)$ and $E^I \times E^I$ with $(E^2)^I$. The natural injection $\iota: G \rightarrow (E^2)^I = E^I \times E^I$ is continuous but is not an embedding, since it is not open onto its image.

Let $D := \{(x, x) : x \in E\} \subseteq E^2$, and

$$\Delta = D^I = \{(x_j, x_j)_{j \in I} : x_j \in E\} \subseteq (E^2)^I \cong E^I \times E^I$$

denote the respective diagonals. Then Δ is a closed subgroup of $(E^2)^I$ and so $\iota^{-1}(\Delta) = D^{(I)}$ is a closed subgroup of G . We shall denote it by Y . This is a noteworthy and perhaps slightly unexpected fact in view of the density of $E^{(I)}$ in E^I . We verify as an exercise that the subgroup Y is not only closed, but even discrete, since $\iota(Y)$ meets trivially every open subgroup $\{0\} \times E^K$ for a cofinite subset $K \subseteq I$.

Now the product E^I is the projective limit of its finite partial products E^F as F ranges through the directed set \mathcal{F} of finite subsets F of I . Accordingly,

$$G \cong E^{(I)} \times \lim_{F \in \mathcal{F}} E^F \cong \lim_{F \in \mathcal{F}} (E^{(I)} \times E^F).$$

Let $D_2 = \{(x_j)_{j \in I} \in E^I : (\exists c \in E)(\forall j \in I) x_j = c\}$. Now we consider the following subgroups of G :

$$\begin{aligned} X &:= E^{(I)} \times \{0\}, \\ Z &:= E^{(I)} \times D_2 \end{aligned}$$

whence $X \subseteq Z$. Then $X \vee Y = \overline{E^{(I)} \times E^I} = G$ and so $(X \vee Y) \cap Z = Z$ on the one side, while $Y \wedge Z = Y$ and so $X \vee (Y \wedge Z) = X \vee Y = G$. Hence $X \vee (Y \wedge Z) \neq (X \vee Y) \wedge Z$. Therefore G is a locally compact abelian nonmodular group.

The example shows that the limit of a projective system of locally compact topologically modular group with proper bonding maps need not be a topologically modular group and that a local product of a collection of finite abelian modular groups may fail likewise to be a topologically modular group.

Locally compact abelian topologically modular groups were classified by MUKHIN in [30]. We now discuss the nonabelian situation.

A first step in the classification is the case of p -groups:

Proposition 9.3. *Let G be a compactly ruled p -group. Then the following statements are equivalent:*

- (1) *G is a topologically modular group.*
- (2) *G is a topologically quasihamiltonian group with a base group A that is a topologically modular locally compact abelian group.*

In contrast, however, with topologically quasihamiltonian groups, the normal p -Sylow subgroups are not the only building blocks of topologically modular groups. There is one additional category of building blocks which in the discrete situation were known since the pioneering work of Iwasawa in the forties of the last century, see e.g. [23].

9.1. Iwasawa (p, q) -Factors.

Example 9.4. For a prime q let \mathbf{A} be an additively written locally compact abelian group of exponent q and be either compact or discrete. Thus, algebraically, \mathbf{A} is a vector space over the field $\text{GF}(q)$.

Now let p be a prime such that $p \mid (q - 1)$. Then the multiplicative group of $\text{GF}(q)$ contains a cyclic subgroup Z of order p . Let $C = \overline{\langle t \rangle}$ be any p -procylic group (that is, $C \cong \mathbb{Z}(p^k)$ for some $k \in \mathbb{N}$ or $C \cong \mathbb{Z}_p$), and let $\psi: C \rightarrow Z$ be an epimorphism. Then C acts on \mathbf{A} via $r * a = \psi(r) \cdot a$. Since Z is of order p , the kernel of ψ is an open subgroup of C of index p .

Set $G = \mathbf{A} \rtimes_{\psi} C$, the semidirect product for the action of C on \mathbf{A} . Then $A := \mathbf{A} \times \{1\}$ is a base subgroup of the near abelian locally compact group G , and $H = \overline{\langle (0, t) \rangle} = \{0\} \times C$ is a procylic scaling p -subgroup.

There are many maximal p -subgroups of G , namely, each $\overline{\langle (a, t) \rangle}$ for any $a \in A$, and there is one unique maximal q -subgroup which is normal, namely, A .

The simplest case arises when we take for C the unique cyclic group $S_p(Z)$ of Z of order p , in which case we have $G \cong \mathbf{A} \rtimes \mathbb{Z}(p)$ and the set of elements of order p is $\mathbf{A} \times (\mathbb{Z}(p) \setminus \{0\})$ and the set of q -elements is $\mathbf{A} \times \{0\}$.

The class of locally compact near abelian topologically quasihamiltonian groups described in Example 9.4 is relevant enough in our classification to deserve a name:

Definition 9.5. A locally compact group G which is isomorphic to a semidirect product $\mathbf{A} \rtimes_{\psi} C$ as described in Example 9.4 will be called an *Iwasawa (p, q) -factor*. The primes p and q are called the *primes of the factor G* .

The prime graph \mathcal{G} of an Iwasawa (p, q) -factor is one sloping edge e_{pq} with its endpoints.

We would like to see an abstract characterisation of a (p, q) -factor. For the purpose of presenting one let us formulate some terminology for an automorphic action $(h, a) \mapsto h \cdot a : H \times A \rightarrow A$ inducing a morphism $\alpha : H \rightarrow \text{Aut}(A)$, $\alpha(h)(a) = h \cdot a$. If $H/\ker \alpha$ is an abelian group of order p for a prime number p , we shall say that

the action of H on A is of order p .

If H is a subgroup of a group G and A is a normal subgroup of G , then H acts on A via $h \cdot a = hah^{-1}$. If this action is of order p , we say that

H induces an action of order p on A .

Proposition 9.6. *Let p and q be primes satisfying $p|(q-1)$. A near abelian group G is an Iwasawa (p, q) -factor if and only if it satisfies the following conditions:*

- (a) *$A = G'$ is an abelian group of exponent q ; it is either compact or discrete subgroup of G ;*
- (b) *There is a scaling group H which is a procyclic p -group; it induces an action of order p on A .*

If these conditions are satisfied, then $G = A \rtimes H$ is a semidirect product and $Z(G) = \{h^p : h \in H\}$.

The significance of the (p, q) -factors for our classification is due to the following fact which requires a technical proof that is not exactly short:

Proposition 9.7. *Let $G = AH$ be an Iwasawa (p, q) -factor and A a topologically modular abelian group. Then G is a topologically modular group.*

Since a nondegenerate (p, q) -factor does not meet the criteria of a topologically quasihamiltonian locally compact group in Theorem 8.4, this allows us to remark a significant difference between topologically quasihamiltonian and topologically modular groups:

Corollary 9.8. *Any nondegenerate Iwasawa (p, q) -factor provides a topologically modular group which is not topologically quasihamiltonian.*

After a thorough discussion of compactly ruled topologically modular groups, using much of the information accumulated on near abelian groups we arrive at the following classification of periodic locally compact topologically modular groups:

Theorem 9.9 (The Main Theorem on Topologically Modular Groups). *Let G be a compactly ruled topologically modular group. Then π is a disjoint union of a set J of sets σ of prime numbers which are either empty, or singleton sets $\sigma = \{p\}$ such that G_σ is a normal p -Sylow subgroup and an Iwasawa p -factor, or two element sets $\{p, q\}$ such that for $p < q$ the set G_σ is a normal σ -Sylow subgroup and an Iwasawa (p, q) -Factor, such that*

$$G = \prod_{\sigma \in J}^{\text{loc}} (G_\sigma, C_\sigma)$$

for a family of compact open subgroups $C_\sigma \subseteq G_\sigma$. In particular, G is a periodic near abelian locally compact group.

Conversely, every near abelian locally compact G of this form is a topologically modular locally compact group.

We notice that, in the prime graph of G , the Sylow p -subgroups G_p constitute the connected components of either isolated vertices or vertical edges with its endpoints, while the Sylow subgroups $G_{\{p,q\}}$ which are Iwasawa (p, q) -components are connected components consisting of sloping edges with their endpoints. Moreover, every prime graph having such connected components can be realised as the prime graph of a periodic locally compact topologically modular group.

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